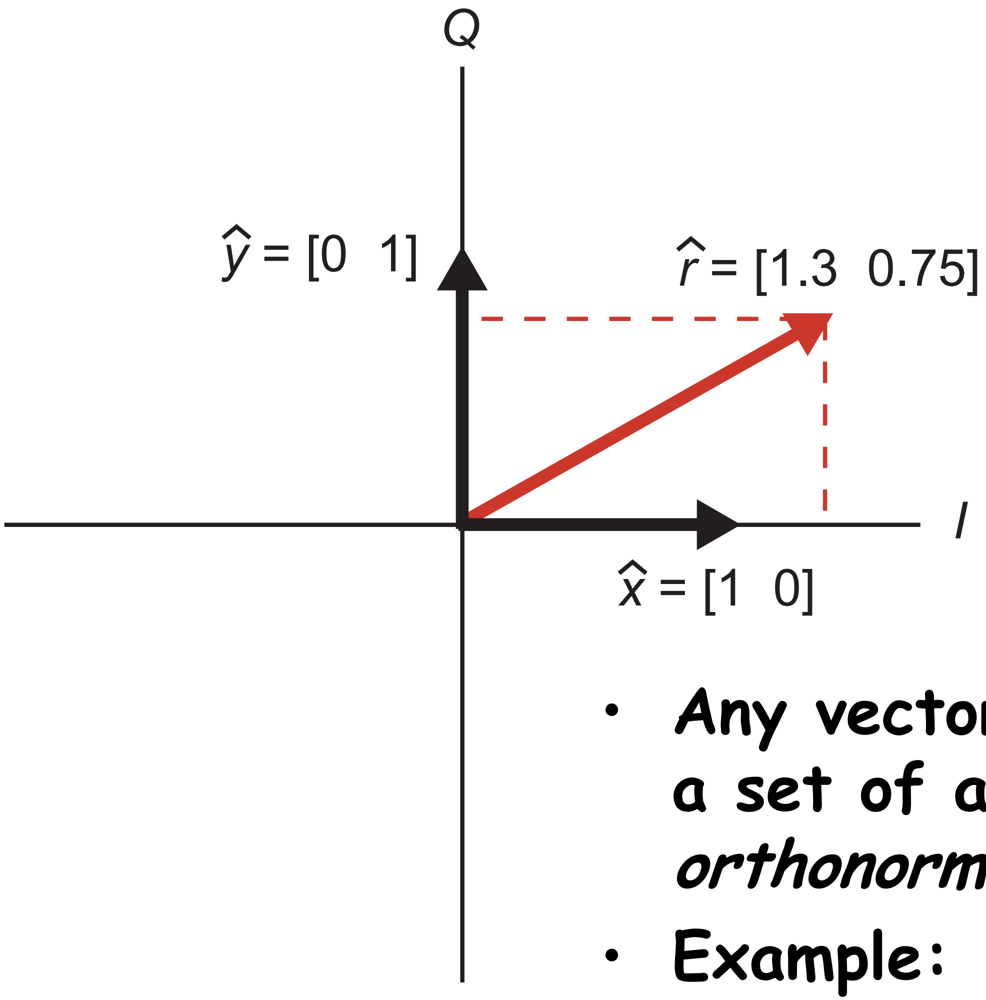
Intro to Fourier Series

- Vector decomposition
- Even and Odd functions
- Fourier Series definition and examples

Review of Vector Decomposition

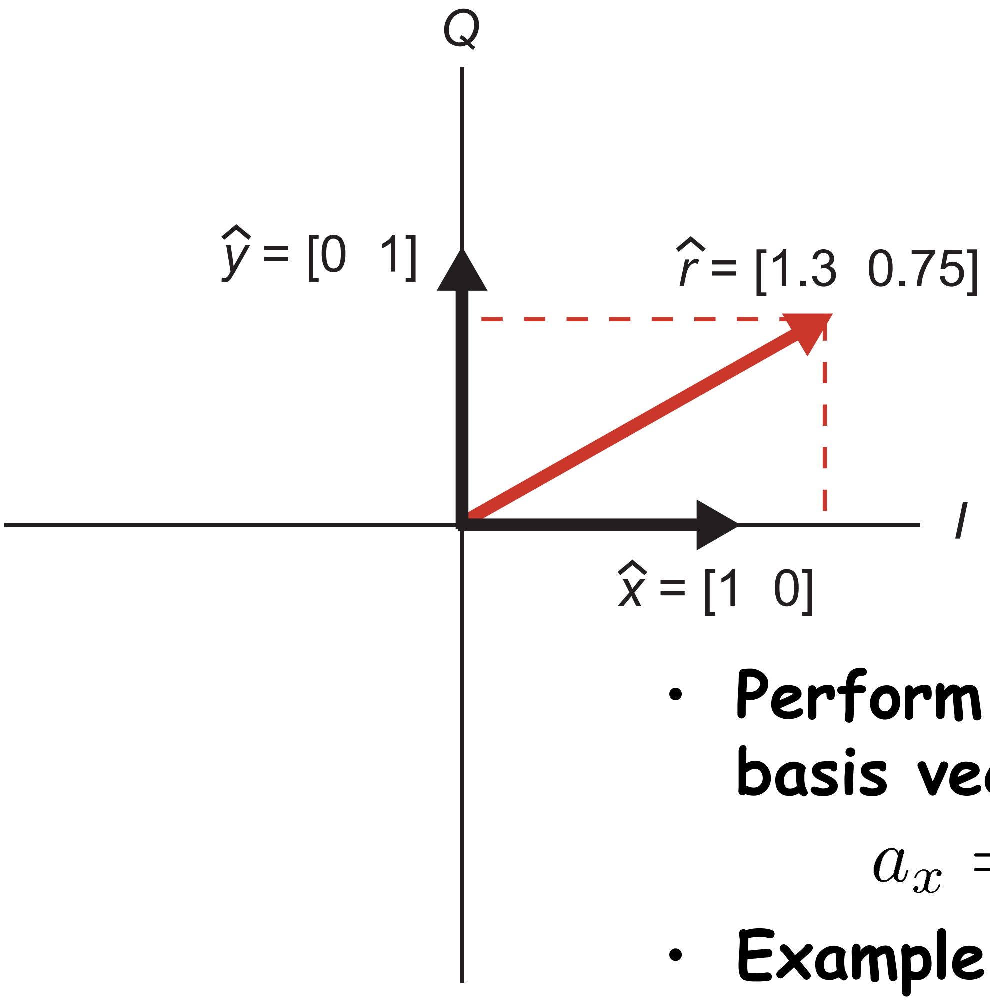


- **Any vector can be decomposed into a set of appropriately weighted *orthonormal basis* vectors**
- **Example:**

$$\hat{r} = a_x \hat{x} + a_y \hat{y}$$

$$a_x = ??, \quad a_y = ??$$

Calculation of Vector Weights



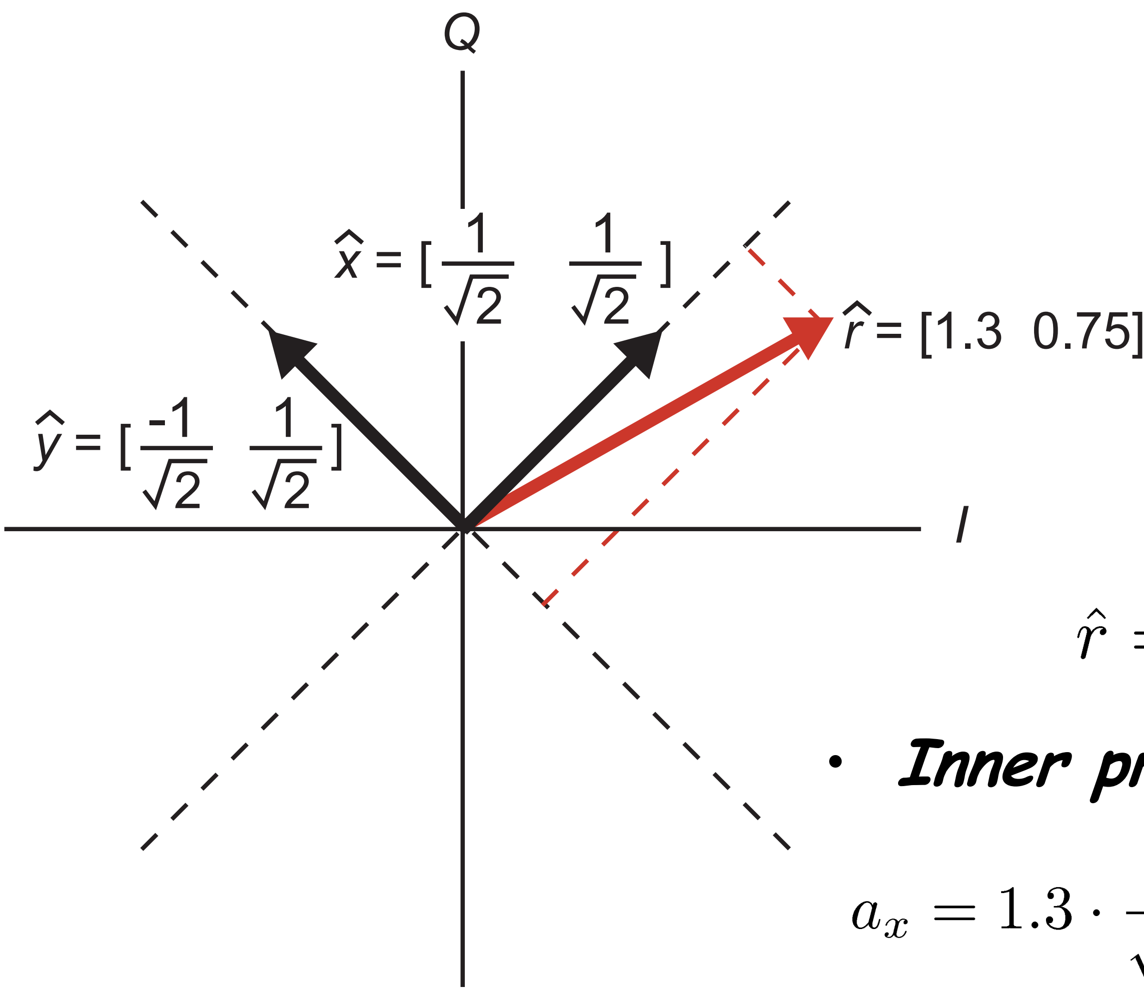
- **Perform *inner products* with basis vectors**

$$a_x = \hat{r} \cdot \hat{x}, \quad a_y = \hat{r} \cdot \hat{y}$$

- **Example:**

$$a_x = 1.3 \cdot 1 + .75 \cdot 0 \ = \ 1.3$$
$$a_y = 1.3 \cdot 0 + .75 \cdot 1 \ = \ 0.75$$

The Basis Vectors are Not Unique



$$\hat{r} = a_x \hat{x} + a_y \hat{y}$$

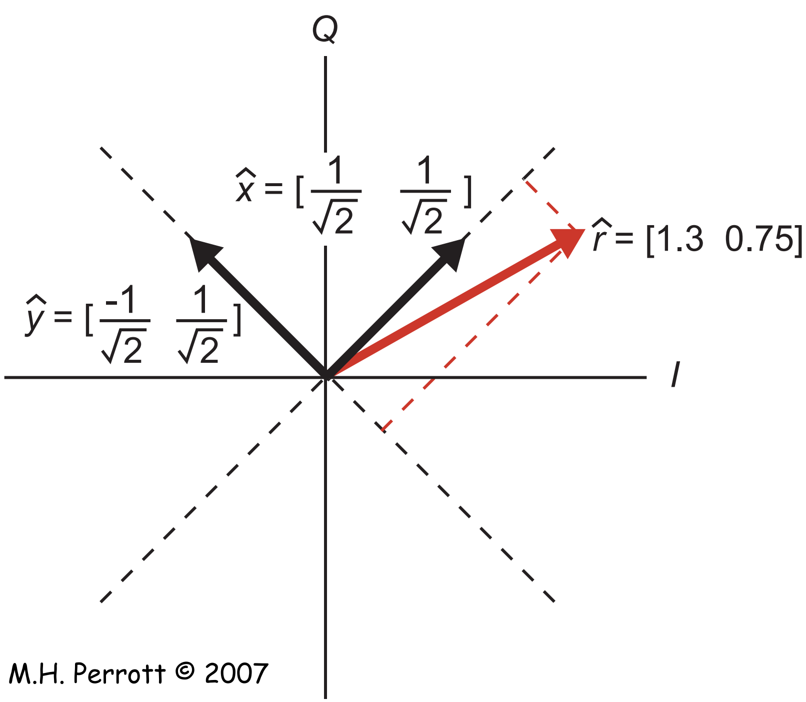
- ***Inner product*** **calculations:**

$$a_x = 1.3 \cdot \frac{1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{2.05}{\sqrt{2}}$$

$$a_y = 1.3 \cdot \frac{-1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{-.55}{\sqrt{2}}$$

Observations on Basis Decomposition

- We can consider any vector as a *sum* of *weighted* orthonormal basis vectors
- The weights are determined by an *inner product* calculations (also known as *projections*)
  - Consist of element-by-element multiplications followed by addition of the resulting products



$$\hat{r} = a_x\hat{x} + a_y\hat{y}$$

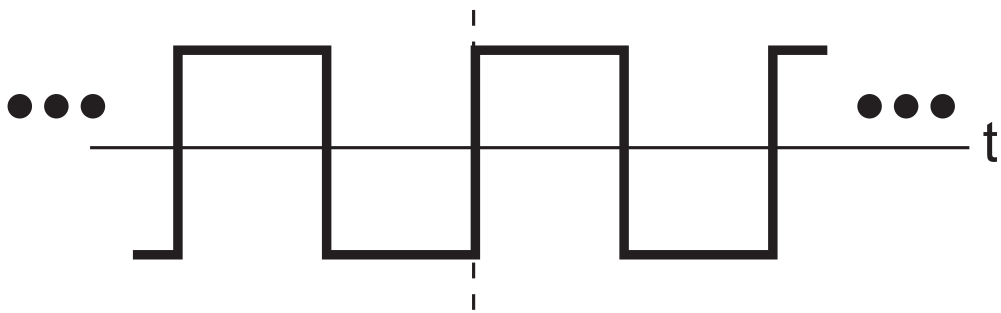
- *Inner product* calculations:

$$a_x = 1.3 \cdot \frac{1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{2.05}{\sqrt{2}}$$

$$a_y = 1.3 \cdot \frac{-1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{-.55}{\sqrt{2}}$$

Can We Decompose *Functions*?

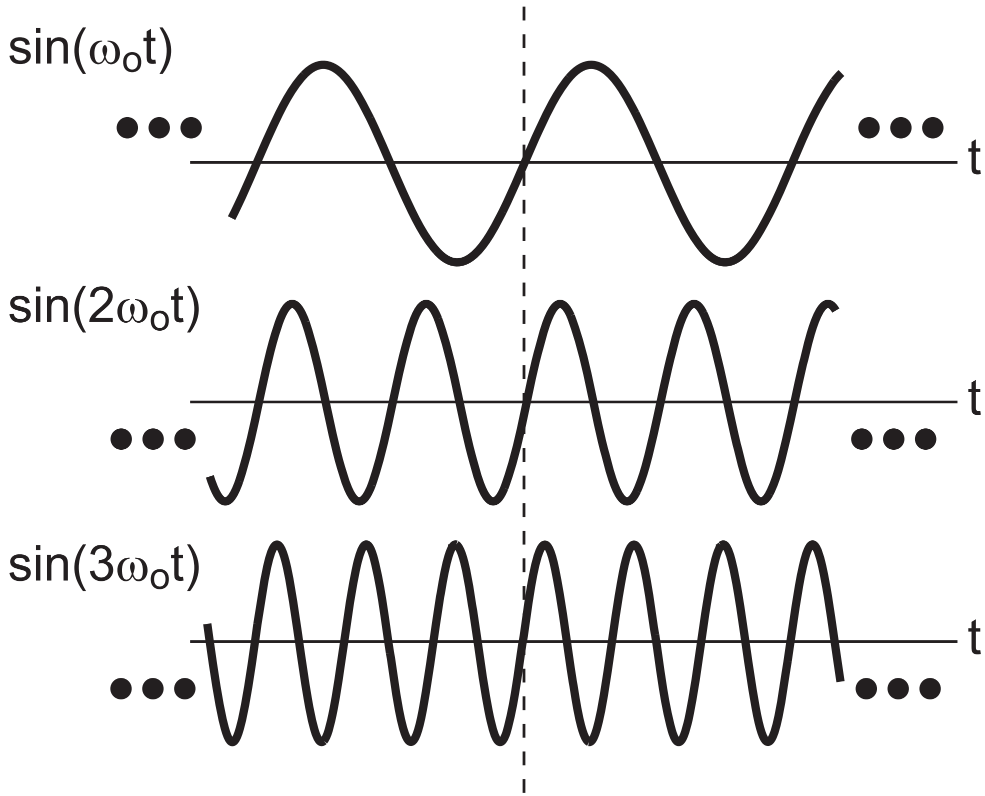
- Consider a periodic function such as a square wave



- Could we decompose the above waveform into a weighted sum of *basis functions*?
- If so, what would be a good choice for such basis functions?
- How would we calculate the weights?

Consider Sine Wave Basis Functions

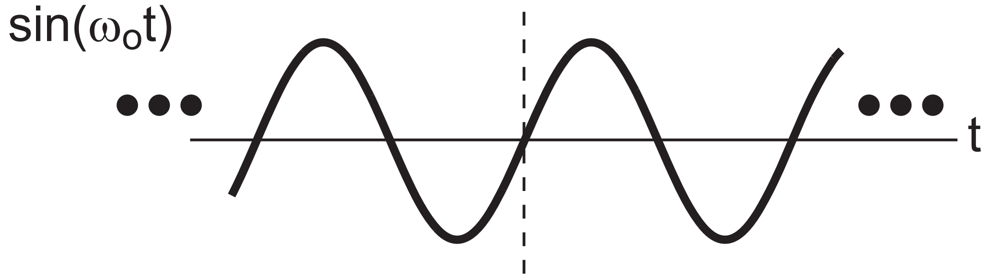
- **Suppose we consider sine waves of progressively increasing frequencies as our basis functions**



- **Check out the following Java applet demo:**
  - **Available at: http://www.falstad.com/fourier/**

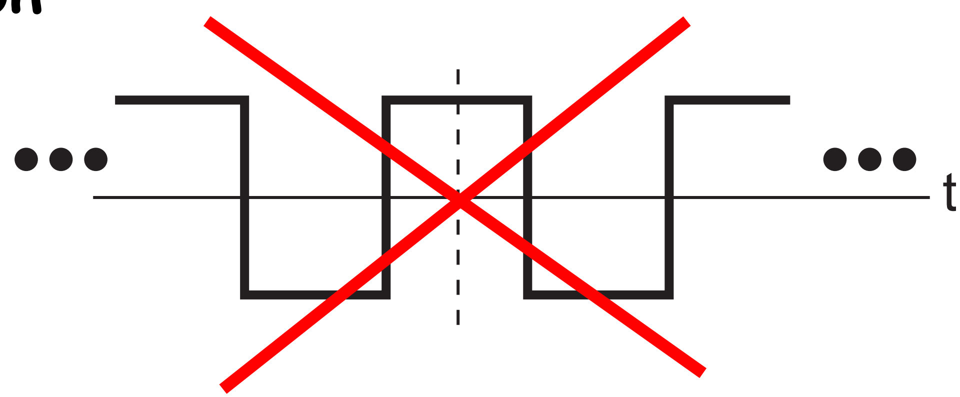
Issue: Sine Waves Are Limited

- A sine wave corresponds to an *odd* function



- Odd function definition: $$f(t) = -f(-t)$$

- Adding odd functions together can *only* produce an odd function

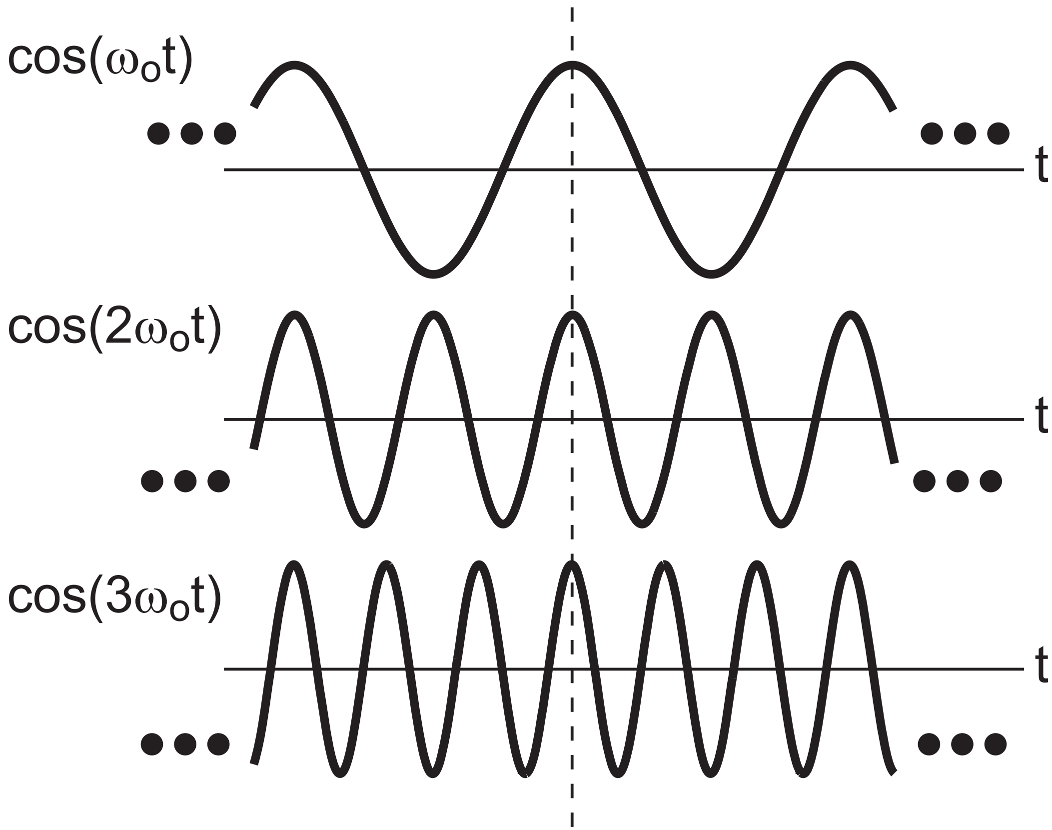


Consider Cosine Wave Basis Functions

- **Even function definition:**

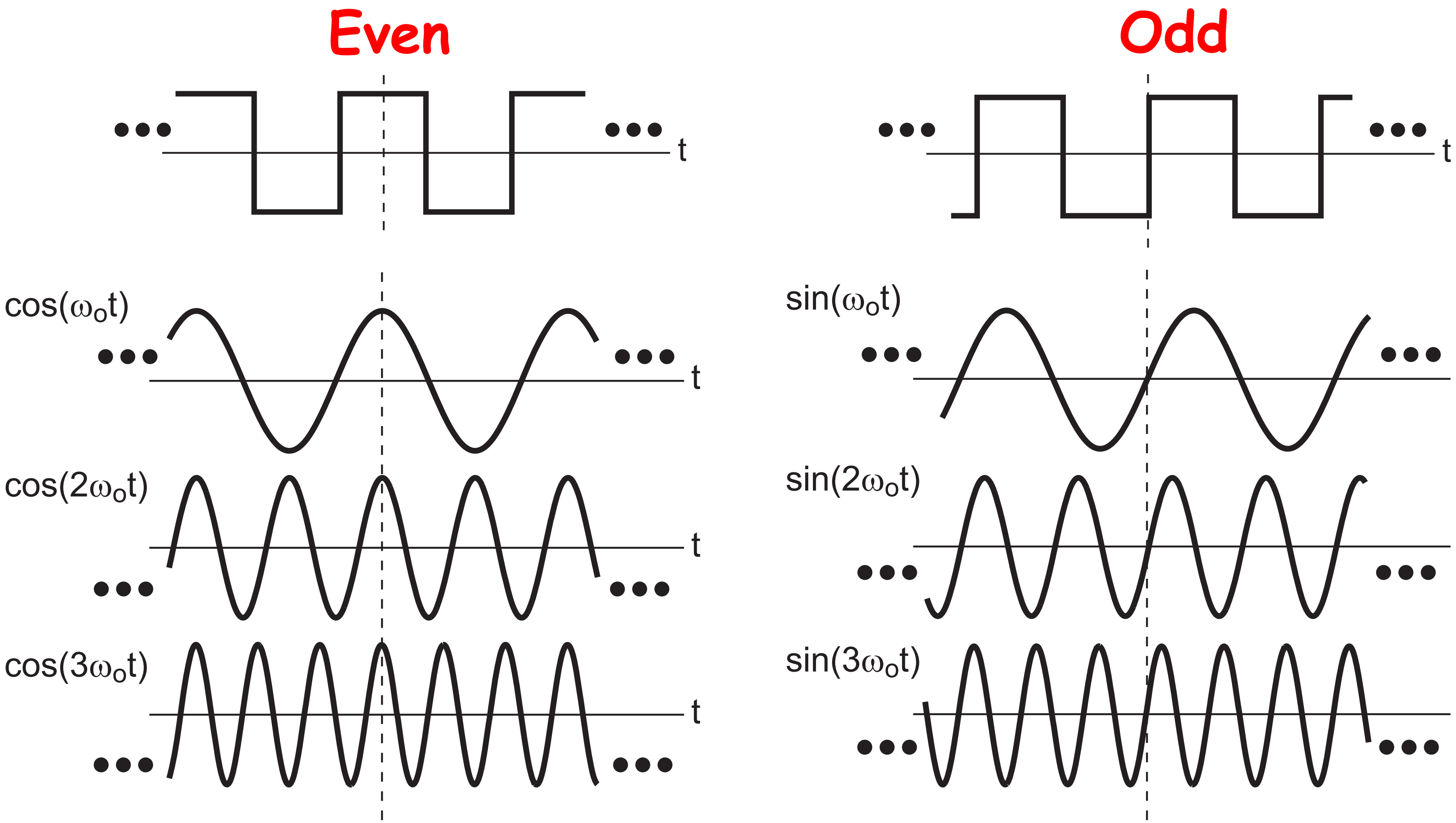
$$f(t) = f(-t)$$

- **Cosine waves are even functions**



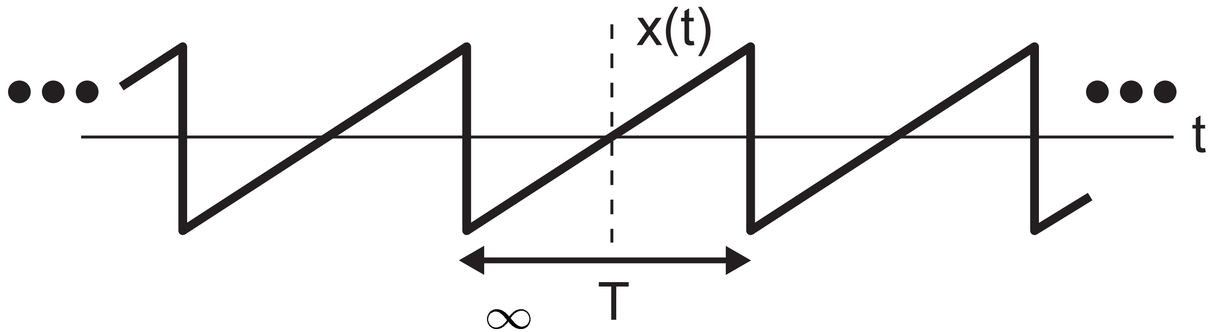
Combine Cosines and Sines

- If we use *both* cosine and sine waveforms as basis functions, we can realize *both* even and odd functions (and any combination)



The Fourier Series

- **A periodic waveform, x(t), with period T can be represented as an infinite sum of weighted cosine and sine waveforms**



$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_o t) + b_n sin(nw_o t)$$

$$\text{where for } n > 0 :$$

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\cos(nw_o t)dt,\ b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\sin(nw_o t)dt$$

$$\text{and where :} \qquad w_o = \frac{2\pi}{T}, \quad a_0 = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt$$

Intuition for Fourier Series

- Compare Fourier Series to vector decomposition:

Vector Decomp.

$$\hat{r} = a_x\hat{x} + a_y\hat{y}$$

$$a_x = \hat{r}\cdot\hat{x},$$

$$a_y = \hat{r}\cdot\hat{y}$$

Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nw_o t) + b_n sin(nw_o t)$$

$$a_0 = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt,$$

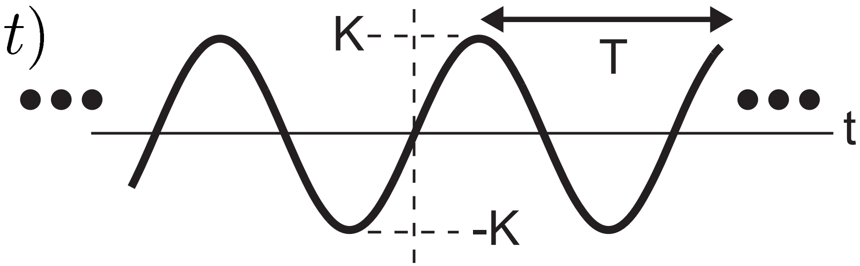
$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\cos(nw_o t)dt,$$

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\sin(nw_o t)dt$$

- The Fourier Series weight (i.e., a_n and b_n) calculations are analogous to vector inner products!

Sine Wave Example

$$x(t) = K\sin(w_o t)$$



$$a_0 = \frac{1}{T}\int_{t_o}^{t_o+T} K\sin(w_o t)dt = \boxed{0}$$

(DC Average is 0)

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} K\sin(w_o t)\cos(nw_o t)dt$$

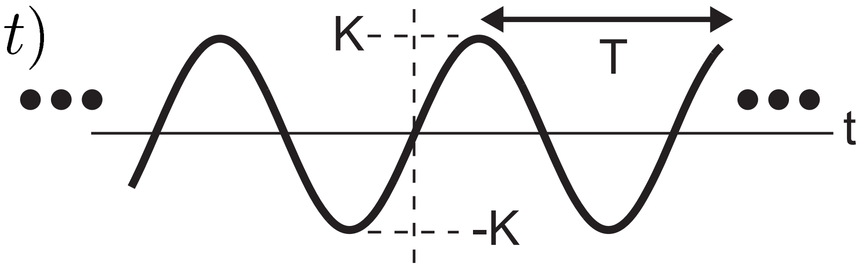
$$= \frac{K}{T}\int_{t_o}^{t_o+T} \sin((n-1)w_o t) + \sin((n+1)w_o t)dt = \boxed{0}$$

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} K\sin(w_o t)\sin(nw_o t)dt$$

$$= \frac{K}{T}\int_{t_o}^{t_o+T} \cos((n-1)w_o t) + \cos((n+1)w_o t)dt = \boxed{\begin{cases} K & (n=1) \\ 0 & (n>1) \end{cases}}$$

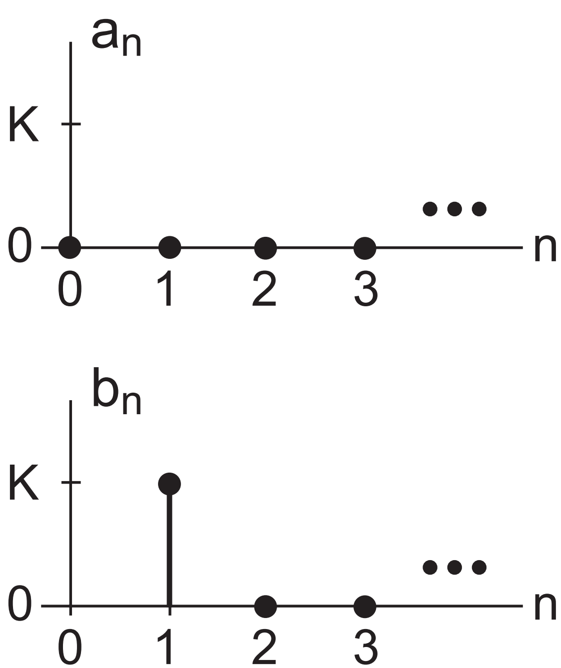
Graphical View of Fourier Series (Sine)

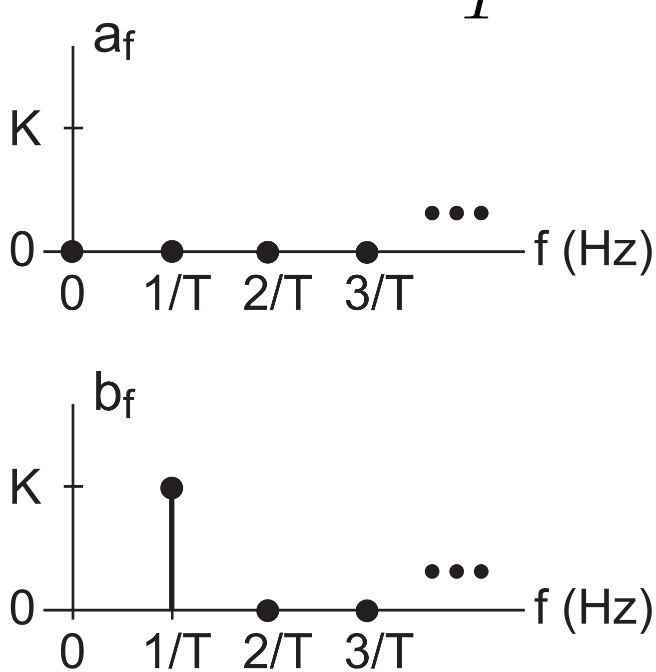
$$x(t) = K\sin(w_o t)$$



- **We can plot Fourier coefficients as a function of index or frequency**

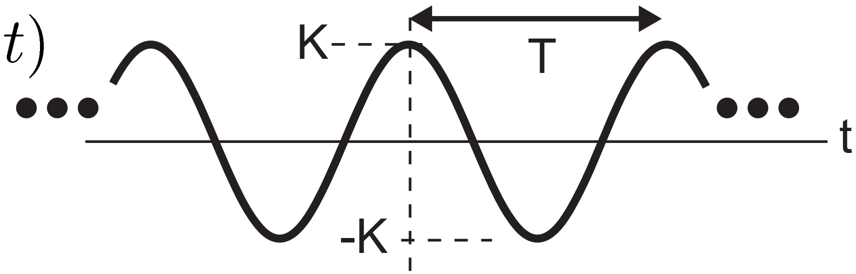
$$\text{Note: } f = \frac{n}{T}$$



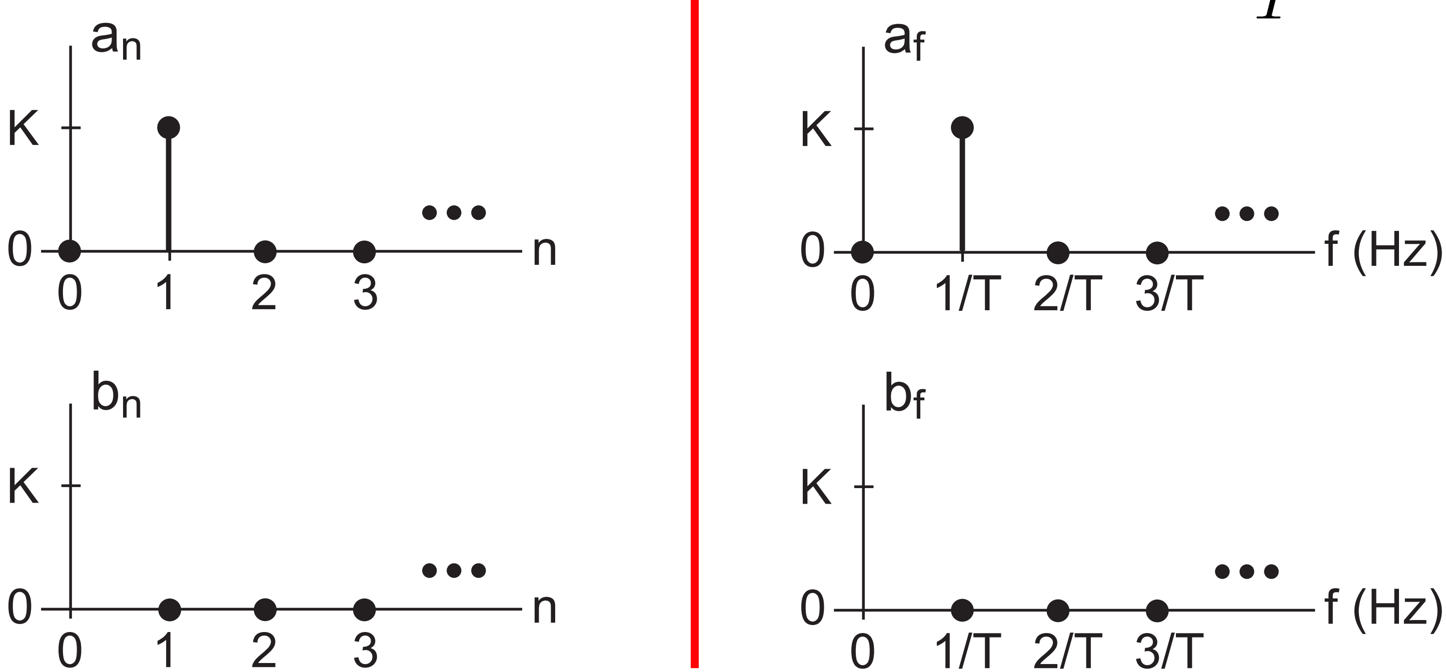


Fourier Series of Cosine

$$x(t) = K\cos(w_o t)$$

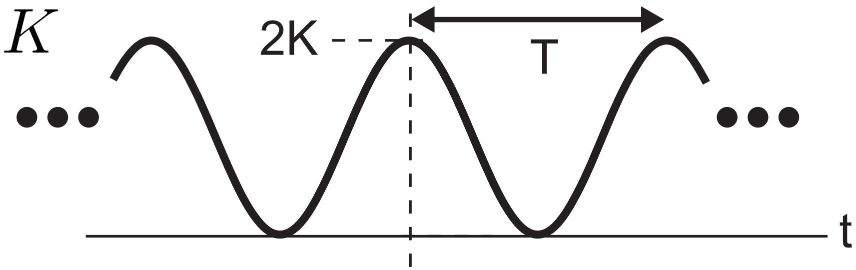


$$\text{Note: } f = \frac{n}{T}$$

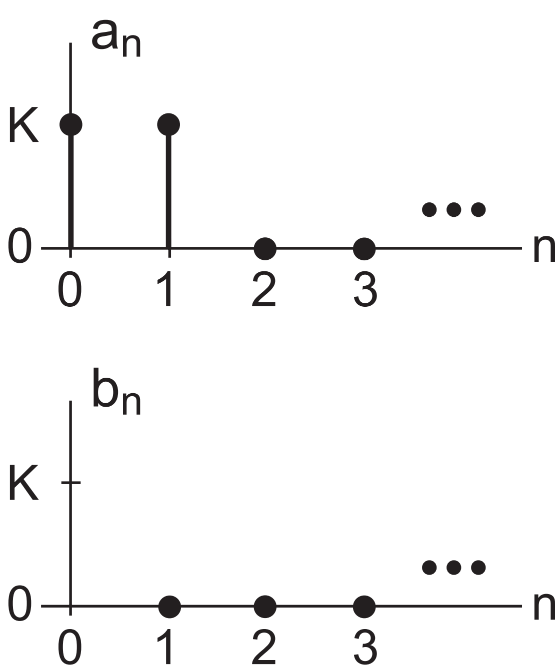


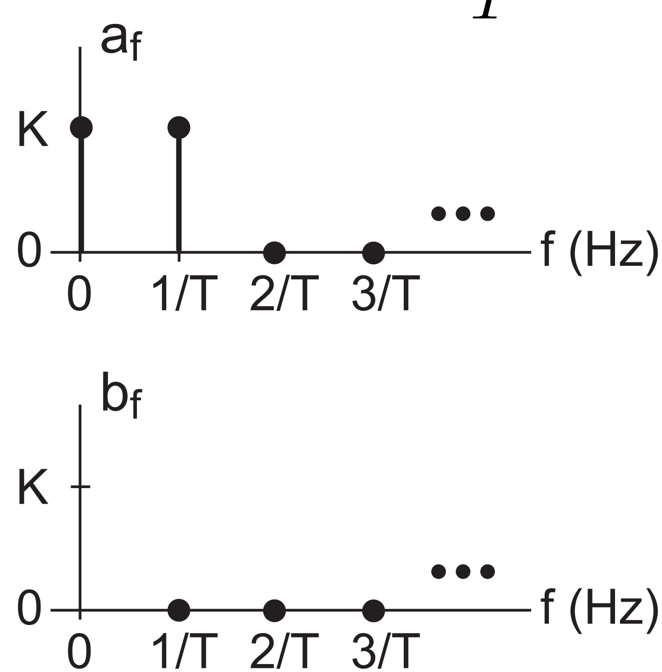
Fourier Series of Cosine with DC component

$$x(t) = K\cos(w_o t) + K$$



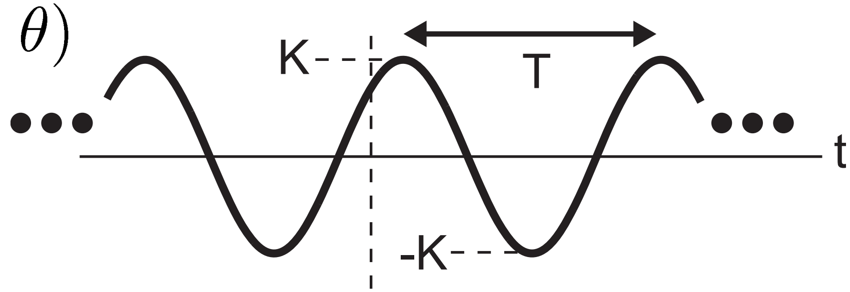
Note: $f = \frac{n}{T}$





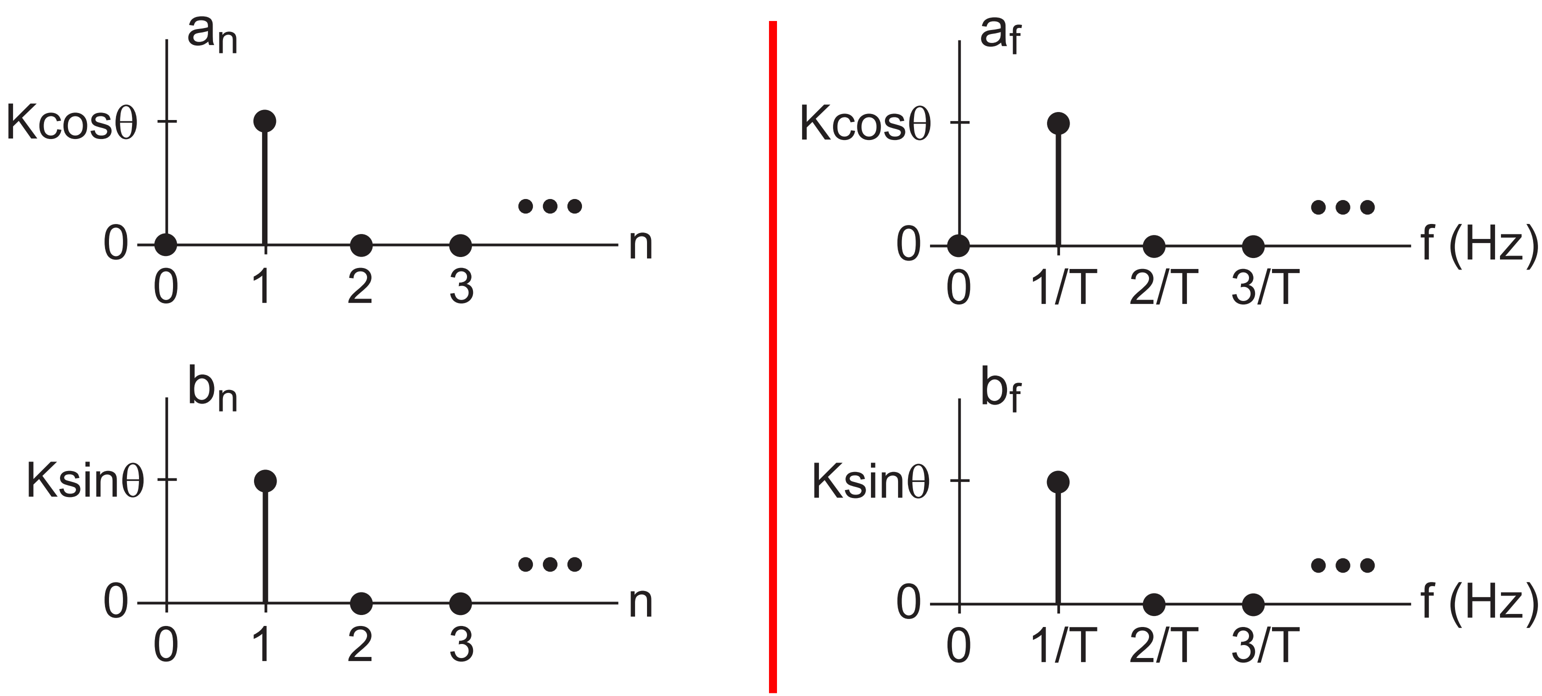
Fourier Series of Phase-Shifted Cosine

$$x(t) = K\cos(w_o t - \theta)$$



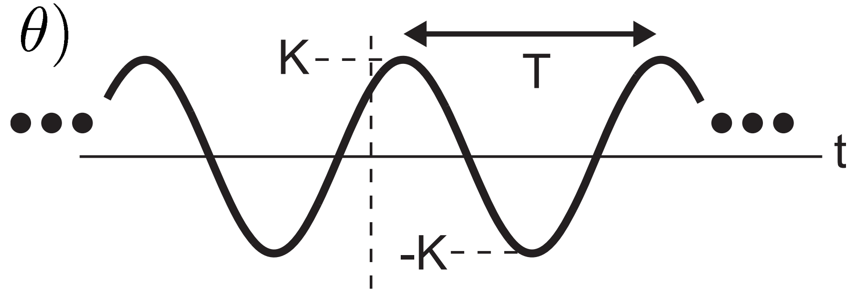
- **Using a well known trigonometric identity:**

$$K\cos(w_o t - \theta) = K\cos(\theta)\cos(w_o t) + K\sin(\theta)\sin(w_o t)$$



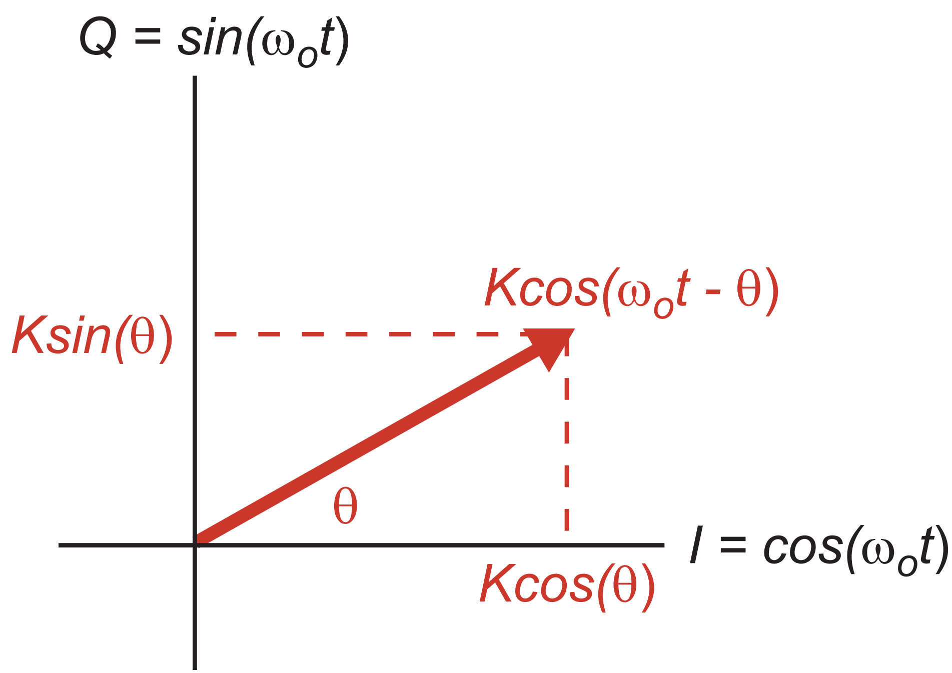
Vector View of Phase-Shifted Cosine

$$x(t) = K\cos(w_o t - \theta)$$

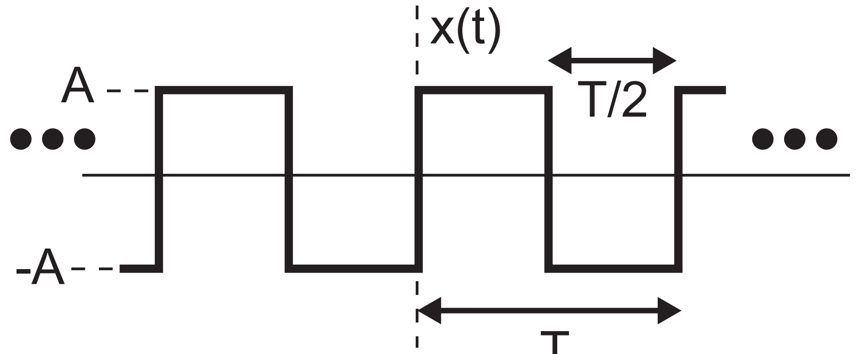


- **Using a well known trigonometric identity:**

$$K\cos(w_o t - \theta) = K\cos(\theta)\cos(w_o t) + K\sin(\theta)\sin(w_o t)$$



Square Wave Example



- **By inspection:**
  - **DC average = 0 , a_0 = 0**
  - **x(t) is odd , a_n = 0 (n ¿ 1)**

$$b_n = \frac{2}{T}\left(\int_{-T/2}^{0} -A\sin(nw_o t)dt + \int_{0}^{T/2} A\sin(nw_o t)dt\right)$$

$$= \frac{4A}{T}\int_0^{T/2} \sin(nw_o t)dt = \frac{4A}{T}\frac{1}{nw_o}(-\cos(nw_o T/2) + 1)$$

$$= \frac{4A}{T}\frac{T}{n2\pi}(-\cos(n\frac{2\pi}{T}T/2) + 1) = \frac{2A}{n\pi}(-\cos(n\pi) + 1)$$

$$\Rightarrow \boxed{b_n = \frac{4A}{n\pi} \text{ for } n \text{ odd}, \quad b_n = 0 \text{ for } n \text{ even}}$$

Summary

- **Vector decomposition provides a nice starting point for understanding Fourier Series**
  - Vector decomposition into a sum of weighted basis vectors
- **Fourier Series decomposes *periodic* waveforms into an infinite sum of weighted cosine and sine functions**
  - We can look at waveforms either in 'time' or 'frequency'
  - Useful tool: even and odd functions
- **Some issues we will deal with next time**
  - Fourier Series definition covered today is not very compact
    - We will look at a simpler formulation based on *complex exponentials*
  - Fourier Series only deals with *periodic* waveforms
    - We will introduce the Fourier Transform to deal with non-periodic waveforms